DIGITAL RECKONING AMONG THE ANCIENTS.

By LEON J. RICHARDSON, University of California.

Primitive man developed his notions of number to no small extent by aid of the fingers. Hence his habit of counting in systems of five and ten. As social relations grew, the needs of communication led to certain uniformities of practice. These signs and arithmetical processes must have differed among tribes; at the same time the fixed elements in the case, namely the ten fingers, made for similarities. Such devices for reckoning, moreover, once hit upon, especially if marked by superior convenience, tended to spread from tribe to tribe and from region to region, just as cleverly devised modes of writing numbers passed with little change from the Egyptians to the Phœnicians and thence to Palmyra and the Syrians.

7

I. Use of the Fingers in Representing Numbers.

The practice of indicating numbers on the fingers was common among ancient Egyptians, Babylonians, Greeks, and Romans. Moreover, the system was eventually developed to such an extent that all numbers from one to 10,000, and sometimes even larger numbers, could be so expressed. References to the subject are often met with in Greek and Latin literature, authors usually taking it for granted that their readers will understand details. Pliny the Elder (*Natural History*, 34, 7, 33) says that "king Numa dedicated a statue of two-faced Janus . . . the fingers being put in a position to show 365 . . . and thus to represent him as the god of time and duration." Macrobius (1, 9, 10), referring to copies of this statue, says the number 300 was "held" in the right hand and 65 in the left. A statue of the philosopher Chrysippus, who devoted much attention to mathematics, had "the fingers drawn together so as to indicate numbers" (Sidonius: *Epist.*, 9, 9, 14). The ring-finger, for example, when bent, indicated the number six (Macrobius, 7, 13, 10). The sign for 500 was made *flexo pollice* (Quintilian, 11, 3, 117). St. Jerome (*Adversus Iovinianum*, 1, 3) informs us that "Thirty is associated with marriage. For the very union of the fingers, as if embracing, . . . depict the husband and wife. Sixty indeed is associated with widows, for the reason that they are placed in straits and misery. . . . The number one hundred however (pray, reader, attend carefully) passes over from the left hand to the right . . . and forming a circle bodies forth the crown of virginity." Cassiodorus—historian, statesman and monk of the sixth century—in commenting on the sixtieth Psalm remarks: "The numbering of this Psalm moreover is not barren of interest. For the number sixty belongs fitly to celibates and widows, being represented by a bound position of the fingers."

The practise of representing numbers on the fingers gave rise to the English word *digit* as a name for each of the numerals below ten. The earliest known instance of this word in our literature, namely in the works of John de Trevisa under date of 1398, takes the form *digitus*, the Latin word having developed that meaning in the late middle ages, notably in books on algorism.

Fortunately we have not been left in the dark as to how the individual numbers were represented, for the complete system is set forth by the Venerable Bede and by Nicolaus Rhabda of Smyrna, both of the eighth century, men dwelling in widely separated parts of the world, one writing in Latin and the other in Greek. Bede's account, which antedates all others that have come down to us, is given in his book *De loquela per gestum digitorum* and Rhabda's is incorporated in the work of Nicolaus Caussinus *De eloquentia sacra et humana* (Paris, 1636). There exist still other accounts, notably one contained in the Persian and Arabic lexicon of Ghiyás and translated into English by E. H. Palmer. These writers set down what had been traditional, doubtless, from early times, for they agree substantially among themselves and their matter harmonizes in the main with the scattered and casual observations of classical authors.

From the foregoing sources we learn that numbers were expressed by the following signs. The hand, unless otherwise stated, was held upward, the palm flat, the fingers together, except the thumb, which did not touch the second finger.

1—5th finger of the left hand bent at the middle joint.
2—4th and 5th fingers bent at the middle joint.
3—3d, 4th and 5th fingers bent at the middle joint.
4—3d and 4th fingers bent at the middle joint.
5—3d finger bent at the middle joint.
6—4th finger bent at the middle joint.
7—5th finger closed on the palm.
8—4th and 5th fingers closed on the palm.
9—3d, 4th and 5th fingers closed on the palm.
10—Tip of the 2d (or index) finger touched the middle joint of the thumb.
11—The signs for 10 and 1 were made coincidentally. The same method applied to numbers from 12 to 19.
20—The thumb was placed between the 2d and 3d fingers in such a way that the thumb nail touched the middle joint of the 2d finger.
30—The thumb and the 2d finger formed a circle.
40—The thumb and 2d finger stood erect and close together.
50—The thumb, bent at both joints, rested on the palm.
60—The 2d finger was bent forward over the thumb, which remained in the position just described.
70—The first joint of the 2d finger rested upon the first joint of the thumb, which was held nearly straight.
80—The tip of the 2d finger rested upon the first joint of the thumb.
90—The thumb was bent over the first joint of the 2d finger.

The signs so far described were all made with the left hand. We now pass to the part played by the other hand. The sign for 100 did not differ from the sign for 10, except that it was made with the right hand. Similarly related were the signs for 200 and 20, 300 and 30 and so on through 900 and 90.

The sign for 1,000 did not differ from the sign for 1, except that it was made with the right hand. Similarly related were the signs for 2,000 and 2, 3,000 and 3 and so on through 9,000 and 9.

The sign for 10,000 was made by laying the left hand on the chest. 20,000, 30,000 and so on through 90,000 were made by touching various parts of the body

with the same hand. (See Bede, 692.) Similarly the signs for 100,000–900,000 were made by touching corresponding parts of the body with the right hand. The sign for 1,000,000 was the hands clasped, the fingers being interlocked.

II. Use of the Fingers in Counting and Reckoning.

Let us now turn to the other phase of the subject, the use of the fingers in counting and reckoning. The Homeric verb πεμπάζειν means "to count on the five fingers," "to count by fives." Herodotus (6, 63) employs the expression ἐπὶ δακτύλων συμβάλλεσθαι meaning "to reckon on the fingers." Significant also in this connection is a passage in Aristophanes (*Vespæ*, 655–657): "Hear then . . . and first of all do an easy sum—not with counters, but with your fingers—the tribute collectively which accrues to us from the cities."

There is a verse in Plautus (*Miles Gloriosus*, 204) which runs: "He reckons the pros and cons on the fingers of the right hand" (*dextera digitis rationem conputat*). To a Roman familiar with current mathematical usages the expression might convey something more than appears in our literal translation. The clue to the subtler meaning may be found in several ancient writings, but is also to be gathered, as it chances, from Sir Thomas Browne's *Pseudodoxia Epidemica* (iv, iv, 186): "On the left [hand] they accounted their digits and articulate numbers unto an hundred, on the right hand hundreds and thousands." Accordingly the "pros and cons" aforesaid, being reckoned on the right hand, are by implication many in number—jocosely represented as a hundred or more. A similar point is made by the poet Juvenal (*Saturnae*, x, 246): "If one has any faith in great Homer, [Nestor] was an instance of life inferior in duration only to the crow's. Happy was he indeed who put off the hour of his death so long and at last begins to count his years on his right hand" (*suos iam dextra conputat annos*).

An important phrase for our purpose is found in a letter that Cicero once wrote to his friend Atticus, the capitalist (*Ad Atticum*, 5, 21, 12–13): "Everybody present exclaimed that nothing was more shameless than Scaptius, who was not satisfied with 12 per cent., compound interest. . . . Lately a decree of the senate has been passed . . . on the subject of creditors fixing the rate at 12 per cent., simple interest. What difference this makes, if I know your skill at reckoning, you have certainly computed." The phrase "if I know your skill at reckoning" is literally "if I know your fingers" (*si tuos digitos novi*).

Ovid (*Fasti*, 3, 123) has the expression: "the fingers by the aid of which we are wont to count" (*digiti per quos numerare solemus*). But one of the most illuminating passages bearing on our subject is found in Quintilian (1, 10, 35): "As to geometry, people admit that attention to it is of advantage in tender years; for they allow that by this study the thinking powers are excited, the intellect sharpened and quickness of perception produced; but they fancy that it is not, like other sciences, profitable after it has been acquired, but only while it is being studied. Such is the common opinion respecting it. Not without reason, however, have the greatest men devoted much attention to this science;

for while geometry comprises numbers and forms, a knowledge of numbers assuredly is necessary not only to a speaker, but to any one taking even the first steps along the path of learning. For pleading cases in court it is very often in request. On these occasions, to say nothing of becoming confused about sums, if a speaker, by any uncertain or awkward movement of the fingers, differs from the accepted mode of calculation, he is thought to be poorly trained."

Pliny the Younger (2, 20, 3), speaking of a man occupied with thoughts about becoming the heir of a rich woman, says "He moves his lips, keeps his fingers going, reckons" (*movet labra, agitat digitos, computat*). Again Suetonius (*Claudius*, 21) makes this observation on the Roman emperor Claudius: "He gave many gladiatorial shows. . . . There was no form of entertainment at which he was more familiar and free, even thrusting out his left hand, like the commons, and counting aloud on his fingers the gold pieces which were paid to the victors" (*aureos . . . voce digitisque numeraret*).

Apuleius (*Apologia*, 89) protests that a certain lady's age has been represented as sixty, when she was really thirty: "If instead of ten you had said thirty years, you might seemingly have erred in the manual sign of the calculation, [in the former case, the tip of the forefinger touching the middle joint of the thumb,] in the latter those fingers forming a circle. Forty, however, is shown by the flat palm—the simplest of manual signs—and you increase that number by half. An error in the manual sign is out of the question, unless, supposing Pudentilla to be thirty, you counted both consuls with each year."

When we come down to the fourth and fifth centuries A. D., we find St. Jerome (*Adversus Iovinianum*, 1, 46) saying: "He shows that wives are wont to be selected more on the basis of wealth than of character and that many are guided, not by their eyes, but by their fingers, in marrying" (*multos non oculis sed digitis uxores ducere*). In the same period St. Augustine (*City of God*, 18, 53) writes: "He puts aside the fingers of the computers (*calculantium digitos*) and orders silence, who says 'It is not for you to know the times, which the Father hath put in His own power.'"

III. The Four Operations Represented by Use of the Fingers.

Enough has been presented to show how common among the ancients was the use of the fingers in dealing with numbers. The illustrations have been drawn for the most part from the Romans, to be sure, but similar material could be found among the Greeks and other peoples. It is worth remarking in this connection that no trace of finger computation of the sort we have described has been found among the Hindus, to whom ultimately we owe our own system of numerals. It is now time to consider how the fingers were employed in mathematical processes. This is not easy to answer in detail. The case of addition, however, is fairly well understood. When a series of numbers was to be added, the fingers at the outset were made to indicate the first number. The second number was then added mentally and the fingers put in a position to indicate the sum. The third number in its turn was added mentally and the fingers

changed to show the new sum. In this way a person proceeded until he arrived at the final sum. (See Marquardt, 1: 99.) The sight of the fingers was an aid in performing the successive steps of the addition. In subtraction the fingers seem to have been similarly used, according to evidence that has reached us. The same may be said of multiplication and division, for the former may be performed by a series of additions and the latter by a series of subtractions.

Concerning the matter of multiplication, some curious evidence has come to light. Dacia, which in ancient geography corresponds in the main to modern Roumania and Transylvania, became a Roman province under Trajan. It was in fact governed from Rome from about 101 to 256 A. D. During that period four military roads were constructed and several forts were built to protect the inhabitants from the incursions of surrounding barbarians. Numerous colonists from Italy settled in the country. The Dacians on their part adopted the religion and language of the conquerors. The Roman occupation, though comparatively short, may still be traced in Greek and Latin literature, in monoliths, inscriptions and coins, as well as in the language and customs of the Roumanians. Interestingly enough, the Walachian peasants, who dwell in southern Roumania, have preserved an old method of multiplication on the fingers. How old the custom is, or whence derived, no one can tell. It can hardly be of local origin, for it is not absolutely unique. It may hark back to oriental or Greek sources, for an algorism manuscript of about 1200 A. D., now preserved at Heidelberg, contains something similar. (See Cantor, 1: 780.) Again it may be of Roman origin. The last theory is supported by the fact that similar usages, probably of Roman origin, have been noted among French peasants. (See Cantor, 1: 447; also Sittl, p. 262.) Moreover, there were current in the middle ages similar mathematical usages which almost surely arose from Roman sources. The method of multiplication just mentioned is as follows:

TABLE I.

1st cycle $(6\cdots10)$ formula: $10(e + e') + cc'$.
2d cycle $(11\cdots15)$ formula: $15(e + e') + cc' + 75$.
3d cycle $(16\cdots20)$ formula: $20(e + e') + cc' + 200$.
4th cycle $(21\cdots25)$ formula: $25(e + e') + cc' + 375$.
5th cycle $(26\cdots30)$ formula: $30(e + e') + cc' + 600$.
. . . .
nth cycle $[5n + 1\cdots5(n + 1)]$ formula: $5(n + 1)(e + e') + cc' + 5^2(n^2 - 1)$.

ABBREVIATIONS:

e = extended fingers of the right hand.
e' = extended fingers of the left hand.
c = closed fingers of the right hand.
c' = closed fingers of the left hand.

To illustrate the foregoing formulæ, take a problem falling within the first cycle. How many are 7×7? Hold up each hand clenched. Extend the right thumb and index finger (= 7). Extend the left thumb and index finger (= 7) [Note that in this cycle the fingers beginning with the thumb have values respectively of 6, 7, 8, 9, 10.] How many fingers are extended? [The thumbs count as fingers.] Four. $4 \times 10 = 40$. How many fingers are closed? Three on the

right hand and three on the left. $3 \times 3 = 9$. Therefore $7 \times 7 = 49$. This process, it will be observed, presupposes a knowledge of the multiplication table in the modern sense through the fives.

Again, how many are 6×8? Hold up each hand clenched. Extend the right thumb (= 6). Extend the left thumb, index finger and middle finger (= 8). How many fingers are extended? Four. $4 \times 10 = 40$. How many fingers are closed? Four on the right hand and two on the left. $4 \times 2 = 8$. Therefore $6 \times 8 = 48$.

Second cycle problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). [In this cycle the fingers have the values of 11, 12, 13, 14, 15.] How many fingers are extended? Five. $5 \times 15 = 75$. How many are closed? Three on the right hand and two on the left. $3 \times 2 = 6$. $75 + 6 + 75 = 156$. Therefore $12 \times 13 = 156$.

In order to multiply two numbers belonging to different cycles, *e. g.*, 9×13, the larger number is so divided as to bring the problem within a single cycle. *E. g.*, $9 \times 13 = (9 \times 6) + (9 \times 7)$.

In actual practice the Walachians do not go beyond the first cycle. The other cycles as here set forth have been made out inferentially.

The formulæ given above may assume a slightly different form:

TABLE II.

1st cycle $(6 \cdots 10)$ formula: $10(e + e') + cc'$.
2d cycle $(11 \cdots 15)$ formula: $10(e + e') + ee' + 100$.
3d cycle $(16 \cdots 20)$ formula: $20(e + e') + cc' + 200$.
4th cycle $(21 \cdots 25)$ formula: $20(e + e') + ee' + 400$.
5th cycle $(26 \cdots 30)$ formula: $30(e + e') + cc' + 600$.

As regards the *n*th cycle: when n is an odd number, the general formula is the same as that given in Table I; when, however, n is an even number, the general formula is:

$$n\text{th cycle } [5n + 1 \cdots 5(n + 1)] \text{ formula: } 5n(e + e') + ee' + 5^2n^2.$$

By way of illustration, take the problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). How many fingers are extended? Five. $5 \times 10 = 50$. How many are extended on each hand? Two on the right hand and three on the left. $2 \times 3 = 6$. $50 + 6 + 100 = 156$.

The modulus 5 is natural under the influence of the decimal system. However, it is readily possible to substitute 4, 6, 10, or even other numbers. The modulus 6 is illustrated in

TABLE III.

1st cycle $(7 \cdots 12)$ formula: $12(e + e') + cc'$.
2d cycle $(13 \cdots 18)$ formula: $18(e + e') + cc' + 108$.
3d cycle $(19 \cdots 24)$ formula: $24(e + e') + cc' + 288$.
4th cycle $(25 \cdots 30)$ formula: $30(e + e') + cc' + 540$.
5th cycle $(31 \cdots 36)$ formula: $36(e + e') + cc' + 864$.
. . . .
*n*th cycle $[6n + 1 \cdots 6(n + 1)]$ formula: $6(n + 1)(e + e') + cc' + 6^2(n^2 - 1)$.

First cycle problem: How many are 8×9? Hold up each hand clenched. Extend the right thumb and index finger (= 8). Extend the left thumb, index finger and middle finger (= 9). [In order to use this table, one must imagine that he has two little fingers on each hand. The fingers then beginning with the thumb have values respectively of 7, 8, 9, 10, 11, 12.] How many fingers are extended? Five. $5 \times 12 = 60$. How many are closed? Four on the right hand and three on the left. $4 \times 3 = 12$. Since $60 + 12 = 72$, $8 \times 9 = 72$.

The foregoing material may be cast in a form analogous to Table II as follows:

TABLE IV:

1st cycle $(7 \cdots 12)$ formula: $12(e + e') + cc'$.
2d cycle $(13 \cdots 18)$ formula: $12(e + e') + ee' + 144$.
3d cycle $(19 \cdots 24)$ formula: $24(e + e') + cc' + 288$.
4th cycle $(25 \cdots 30)$ formula: $24(e + e') + ee' + 576$.
5th cycle $(31 \cdots 36)$ formula: $36(e + e') + cc' + 864$.

The general formula as given in Table III is here valid, when n is an odd number; when, however, n is an even number, the formula is:

$$n\text{th cycle } [6n + 1 \cdots 6(n + 1)] \text{ formula: } 6n(e + e') + ee' + 6^2n^2.$$

Latin passages similar to those discussed above are as follows:

Plautus: *Stich.*, 706. Ovid: *Pont.*, 2, 3, 117. Seneca: *Epist.*, 88, 10. Pliny: *Natural History*, 2, 87. Irenæus: 5, 30, 1. Tertullian: *Apologeticus*, 19. Martianus Capella: 2, 102; 7, 729 and 746. Firmicus: *Mathesis*, 1, 4, 13. Ambrose: *Tob.*, 7, 25. Pacianus: *Epist. ad Sym. tertia*, 3, 25. Macrobius: *Saturnalia*, 1, 1, 6; *Somnium*, 2, 11, 17. Augustine: *Sermo.* (ed. A. Mai; nova patrum bibliotheca, I), 158, 14; 270, 7. Cassianus: *Conl.*, 24, 26, 7. Boethius: *In Porphy. Comm.*, Sec. 1, 2, p. 138, 19 (Migne, 64).

BIBLIOGRAPHY.

Bedae Venerabilis opera. Pars I. Sect. II. *De loquela per gestum digitorum.* (Migne, vol. 90, p. 686.)

N. Rhabda apud N. Caussinus *De eloquentia sacra et humana* (Paris, 1636).

Rödiger. *Über die im Orient gebräuchliche Fingersprache* (1845).

Friedlein, G. *Über das elementare Rechnen bei den Römern.* Jahrbücher f. Cl. Philologie (Fleckeisen's), 93 (1866): 569.

Palmer, E. H. *Explanation of a difficult passage in Firdausi.* Jour. of Philology (Macmillan), II: 247.

Pick, D., in Zeitschr. f. math. u. naturw. Unterricht (Hoffmann's), 5 (1874): 57.

Stoy, H. *Zur Geschichte des Rechnenunterrichtes* (Jena, 1876). I: 33.

Cantor, M. *Vorlesungen über Geschichte der Mathematik* (Leipzig, 1880), I: 108, 446 and 780.

Marquardt, J. *Das Privatleben der Römer* (Leipzig, 1886²), p. 98.

Sittl, C. *Die Gebärden der Greichen und Römer* (Leipzig, 1890), p. 252.

Bechtel, E. A. *Finger-Counting among the Romans* in the *Fourth Century,* in Classical Philology, January, 1909.

Smith, Wm. R. The article *Numeral* in the Encyclopaedia Britannica (London, 1910).

Rubenbauer: The article *digitus* in Thesaurus Linguae Latinae (Leipzig, 1913).